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A System of First-order Three-Point BVPs at Resonance

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Abstract

This paper deals with existence of solutions to three-point BVPs in perturbed systems of first-order ordinary differential equations at resonance. An existence theorem is established by using the Theorem of Borsuk and some examples are given to illustrate it.

Keywords: *Three-point Boundary Value Problems, Theorem of Borsuk, Resonance Case.*

1 Introduction

In this paper, we investigate the degenerate case of three-point BVPs with linear boundary conditions using the methods and results of Cronin [1, 2]. We obtain the existence of solutions of three-point BVPs at resonance for general BVPs where the parameter $\alpha = \sqrt{2}$ in the interval $[0, 2\pi]$ with rank $\mathcal{L} = 1 < 2$, and rank $\mathcal{L} = 0$ in the interval $[0, 1]$. These results generalize the degenerate case of periodic BVPs considered by Cronin [1, 2], and also the degenerate case of three-point BVP [4].

Consider

$$x' - A(t)x = H(t, x, \varepsilon) = \varepsilon F(t, x, \varepsilon) + E(t), \ 0 \le t \le 1, \tag{1}$$

$$Mx(0) + Nx(\eta) + Rx(1) = 0, \tag{2}$$

where M, N and R are constant square matrices of order n, $A(t)$ is an $n \times n$ matrix with continuous entries, $E : [0, 1] \to \mathbb{R}$ continuous, $F : [0, 1] \times \mathbb{R}^n \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ is a continuous function and $\varepsilon \in \mathbb{R}$ such that $\mid \varepsilon \mid < \varepsilon_0$, and $\eta \in (0, 1)$.

We make use of the Theorem of Borsuk to show the existence of solutions of the BVP (1), (2) under suitable assumptions on the coefficients. Then we present a result for computing the degree of $\psi_0(c) = (\psi_0^1(c_1, c_2), \psi_0^2(c_1, c_2))$ at $(0, 0)$ where the $\psi_0(c_1, c_2)$ are polynomials whose terms of highest order have no common real linear factors; see Cronin [2] p. 296-297. This result is for homogeneous polynomials in two variables which not be odd functions while Borsuk's Theorem holds for continuous odd functions in any dimensions.

2 Preliminaries

We recall the following results of [4].

Lemma 2.1. *[4] Consider the system*

$$x' = A(t)x \tag{3}$$

where $A(t)$ is an $n \times n$ matrix with continuous entries on the interval [0,1]. Let $Y(t)$ be a fundamental matrix of (3). Then the solution of (3) which satisfies the initial condition

$$x(0) = c \tag{4}$$

is $x(t) = Y(t)Y^{-1}(0)c$ where c is a constant n-vector. Abbreviate $Y(t)Y^{-1}(0)$ to $Y_0(t)$. Thus $x(t) = Y_0(t)c$.

Lemma 2.2. *[4] Let $Y(t)$ be a fundamental matrix of (3). Then any solution of (1) and (4) can be written as*

$$x(t, c, \varepsilon) = x(t) = Y_0(t)c + \int_0^t Y(t)Y^{-1}(s)H(s, x(s), \varepsilon)ds. \tag{5}$$

The solution (1) satisfies the boundary conditions (2) if and only if

$$\mathcal{L}c = \varepsilon\mathcal{N}(c, \alpha, \eta, \varepsilon) + d \tag{6}$$

where $\mathcal{L} = M + NY_0(\eta) + RY_0(1)$,
$\mathcal{N}(c, \alpha, \eta, \varepsilon) = -\Big(\int_0^\eta NY(\eta)Y^{-1}(s)F(s, x(s, c, \varepsilon), \varepsilon)ds$
$+ \int_0^1 RY(1)Y^{-1}(s)F(s, x(s, c, \varepsilon), \varepsilon)ds - g(c, x(\eta), x(1))\Big)$,
$d = -\Big(\int_0^\eta NY(\eta)Y^{-1}(s)E(s)ds + \int_0^1 RY(1)Y^{-1}(s)E(s)ds - \ell\Big)$, *and* $x(t, c, \varepsilon)$ *is the solution of (1) given* $x(0) = c$.

Thus (6) is a system of n real equations in $\varepsilon, c_1, \cdots, c_n$ where $c_1, \cdots, c_n$ are the components of c. The system (6) is sometimes called the branching equations.

Next we suppose that $\mathcal{L}$ is a singular matrix. This is sometimes called the resonance case or degenerate case. Now we consider the case rank $\mathcal{L} = n - r$, $0 < n - r < n$. Let E_r denote the null space of $\mathcal{L}$ and let E_{n-r} denote the complement in $\mathbb{R}^n$ of E_r, i.e.

$$\mathbb{R}^n = E_{n-r} \oplus E_r (\text{direct sum}).$$

Let $x_1, \cdots, x_n$ be a basis for $\mathbb{R}^n$ such that $x_1, \cdots, x_r$ is a basis for E_r, and $x_{r+1}, \cdots, x_n$ a basis for E_{n-r}.

Let P_r be the matrix projection onto Ker $\mathcal{L} = E_r$, and $P_{n-r} = I - P_r$, where I is the identity matrix. Thus P_{n-r} is a projection onto the complementary space E_{n-r} of E_r, and

$$P_r^2 = P_r,\ P_{n-r}^2 = P_{n-r} \text{ and } P_{n-r}P_r = P_rP_{n-r} = 0. \tag{7}$$

Without loss of generality, we may assume

$$P_rc = (c_1, \cdots, c_r, 0, \cdots, 0) \text{ and } P_{n-r}c = (0, \cdots, 0, c_{r+1}, \cdots, c_n). \tag{8}$$

We will identify P_rc with $c^r = (c_1, \cdots, c_r)$ and $P_{n-r}c$ with $c^{n-r} = (c_{r+1}, \cdots, c_n)$ whenever it is convenient to do so.

Let H be a nonsingular $n \times n$ matrix satisfying

$$H\mathcal{L} = P_{n-r}. \tag{9}$$

Matrix H can be computed easily. The nature of the solutions of the branching equations depends heavily on the rank of the matrix $\mathcal{L}$.

Lemma 2.3. *[4] The matrix* $\mathcal{L}$ *has rank* $n - r$ *if and only if the three-point BVP (3) and* $Mx(0) + Nx(\eta) + Rx(1) = 0$ *has exactly* r *linearly independent solutions.*

Next we give a necessary and sufficient condition for the existence of solutions of $x(t, c, \varepsilon)$ of three-point BVPs for $\varepsilon > 0$ such that the solution satisfies $x(0) = c$ where $c = c(\varepsilon)$ for suitable $c(\varepsilon)$.

We need to solve (6) for c when ε is sufficiently small. The problem of finding solutions to (1) and (2) is reduced to that of solving the branching equations (6) for c as function of ε for $\mid \varepsilon \mid < \varepsilon_0$. So consider (6) which is equivalent to

$$\mathcal{L}(P_r + P_{n-r})c = \varepsilon \mathcal{N}((P_r + P_{n-r})c, \alpha, \eta, \varepsilon) + d.$$

Multiplying (6) by the matrix H and using (9), we have

$$P_{n-r}c = \varepsilon H\mathcal{N}((P_r + P_{n-r})c, \alpha, \eta, \varepsilon) + Hd, \tag{10}$$

where $H\mathcal{N}((P_r + P_{n-r})c, \alpha, \eta, \varepsilon) = -H\Big(\int_0^\eta NY(\eta)Y^{-1}(s)F(s, x(s, c, \varepsilon), \varepsilon)ds$ $+ \int_0^1 RY(1)Y^{-1}(s)F(s, x(s, c, \varepsilon), \varepsilon)ds - g(c, x(\eta), x(1))\Big)$ and
$Hd = -H\Big(\int_0^\eta NY(\eta)Y^{-1}(s)E(s)ds + \int_0^1 RY(1)Y^{-1}(s)E(s)ds - \ell\Big)$.

Since the matrix H is nonsingular, solving (6) for c is equivalent to solving (10) for c. The following theorem due to Cronin [1, 2] gives a necessary condition for the existence of solutions to the BVP (1) and (2).

Theorem 2.4. *[4] A necessary condition that (10) can be solved for c, with $\mid \varepsilon \mid < \varepsilon_0$, for some $\varepsilon_0 > 0$ is $P_rHd = 0$.*

Definition 2.5. [4] Let E_r denote the null space of $\mathcal{L}$ and let E_{n-r} denote the complement in $\mathbb{R}^n$ of E_r. Let P_r be the matrix projection onto $Ker\ \mathcal{L} = E_r$, and $P_{n-r} = I - P_r$, where I is the identity matrix. Thus P_{n-r} is a projection onto the complementary space E_{n-r} of E_r. If E_{n-r} is properly contained in $\mathbb{R}^n$ then E_r is an $r-$dimensional vector space where $0 < r < n$. If $c = (c_1, \cdots, c_n)$, let $P_rc = c^r$ and $P_{n-r} = c^{n-r}$, then define a continuous mapping $\Phi_\varepsilon : \mathbb{R}^r \to \mathbb{R}^r$, given by

$$\Phi_\varepsilon(c_1, \cdots, c_r) = P_rH\mathcal{N}(c^r \oplus c^{n-r}(c^r, \varepsilon), \alpha, \eta, \varepsilon). \tag{11}$$

where $c^{n-r}(c^r, \varepsilon) = c^{n-r}$ is a differentiable function of c^r and ε. By abuse of notation we will identify P_rc and c^r when convenient and were the meaning is clear from the context so that in defining Φ_ε above from the context we interpreted $P_rH\mathcal{N}$ as $(H\mathcal{N}_1, \cdots, H\mathcal{N}_r)$. Similarly we will sometimes identify $P_{n-r}c$ and c^{n-r}. Setting $\varepsilon = 0$, we have

$$\Phi_0(c_1, \cdots, c_r) = P_rH\mathcal{N}(c^r \oplus P_{n-r}Hd, \alpha, \eta, 0)$$

where $c^{n-r}(c^r, 0) = P_{n-r}Hd$; note that from the context $c^{n-r}(c^r, 0) = P_{n-r}Hd$ is interpreted as $c^{n-r}(c^r, 0) = (Hd_{r+1}, \cdots, Hd_n)$.

If $E_r = \mathbb{R}^n$ and $P_r = I$, then $P_{n-r} = 0$. Since $P_{n-r} = 0$ it follows that the matrix H is the identity matrix. Thus define a continuous mapping $\Phi_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$, given by $\Phi_\varepsilon(c) = \mathcal{N}(c, \alpha, \eta, \varepsilon)$. Setting $\varepsilon = 0$, we have $\Phi_0(c) = \mathcal{N}(c, \alpha, \eta, 0)$.

3 Main Results

Now we state the Theorem of Borsuk (Piccinini, Stampacchia and Vidossich [5] p. 211).

Theorem 3.1. *If $B_k \subseteq \mathbb{R}^n$ is a bounded open set that is symmetrical with respect to the origin (that is $B_k = -B_k$) and contains the origin. If $\Phi_0 : \bar{B}_k \to \mathbb{R}^n$ is continuous, antipodal, that is*

$$\Phi_0(c) = -\Phi_0(-c)$$

and $0 \notin \Phi_0(\partial B_k)$, then $d(\Phi_0, B_k, 0)$ is an odd number (and thus nonzero).

Next we introduce the computation of the topological degree of a mapping in Euclidean 2-space defined by homogeneous polynomials. The methods and notations described below come from Cronin [2, 3]. Let

$$\Phi_0^1(c_1, c_2) = C_1 \prod_{i=1}^{n} (c_1 - a_i c_2)^{p_i},$$
$$\Phi_0^2(c_1, c_2) = C_2 \prod_{j=1}^{m} (c_1 - b_j c_2)^{p_j},$$

where C_1, C_2, a_i and b_j are real constants. We may disregard C_1, C_2 since they only affect the sign of the topological degree. The topological degree is resolved by examining the changes of sign of $\Phi_0^1(c_1, c_2)$ and $\Phi_0^2(c_1, c_2)$ as c_1, c_2 varies over the boundary of the ball B_k with centre at the origin and arbitrary radius when computing the topological degree of (Φ_0^1, Φ_0^2). We may omit the following factors since none of them affect the degree of (Φ_0^1, Φ_0^2) on B_k at 0.

1. Factors $(c_1 - a_i c_2)$ and $(c_1 - b_j c_2)$ where a_i and b_j have complex conjugates in Φ_0^1, respectively, Φ_0^2.
2. Factors $(c_1 - a_i c_2)$ or $(c_1 - b_j c_2)$ which appear with even exponents where a_i and b_j are real.
3. Factors $(c_1 - a_i c_2)$ and $(c_1 - a_{i+1} c_2)$, if there exists a pair a_i, a_{i+1} $(i < i + 1)$ such that no b_j lies between them (i.e., there is no b_j such that $a_i < b_j < a_{i+1}$). Similarly for pairs b_j, b_{j+1}.
4. Factors $(c_1 - a_r c_2)$ and $(c_1 - a_s c_2)$, if a_r and a_s are the smallest and largest of the array of numbers $a_1, \cdots, a_n, b_1, \cdots, b_m$. Similarly factors $(c_1 - b_r c_2)$ and $(c_1 - b_s c_2)$, if b_r and b_s are the smallest and largest of the array of numbers $a_1, \cdots, a_n, b_1, \cdots, b_m$.

Now, if there are no remaining factors in Φ_0^1 or Φ_0^2, then the topological degree is zero. Otherwise, if we assume that the terms of highest degree of

$\Phi_0^1(c_1, c_2)$ and $\Phi_0^2(c_1, c_2)$ have no common real linear factors after reduction using the conditions 1, 2, 3, and 4 above, then the remaining factors consist of numbers $a_1, \cdots, a_n, b_1, \cdots, b_m$ such that all the $a's$ and $b's$ are distinct values, in which the subscript labelling with either ordering

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_p < b_p$$
$$\text{or}$$
$$b_1 < a_1 < b_2 < a_2 < \cdots < b_p < a_p$$

for some integer $p \leq \min\{m, n\}$. In the first case the degree is p, while in the second case the degree is $-p$.

We now state the second main theorem in this paper (see Cronin [2] p. 38-40).

Theorem 3.2. *Let $\Phi_0(c) = (\Phi_0^1(c_1, c_2), \Phi_0^2(c_1, c_2))$ where $\Phi_0^1(c_1, c_2)$ and $\Phi_0^2(c_1, c_2)$ are polynomials in c_1 and c_2. If we assume that the terms of highest degree of $\Phi_0^1(c_1, c_2)$ and $\Phi_0^2(c_1, c_2)$ are homogenous polynomials with no common real linear factors after reduction using the conditions 1, 2, 3, and 4 above, then*

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_p < b_p$$
$$\textit{or}$$
$$b_1 < a_1 < b_2 < a_2 < \cdots < b_p < a_p$$

for some integer $p \leq min\{m, n\}$. In the first case the degree is p, while in the second case the degree is $-p$. Hence

$$d(\Phi_0, B_k, 0) \neq 0$$

for B_k, a ball with centre at the origin and sufficiently large radius. Then for sufficiently small ε, $\mid \varepsilon \mid < \varepsilon_0$

$$d(\Phi_\varepsilon, B_k, 0) = d(\Phi_0, B_k, 0) \neq 0.$$

Hence there is a solution $x(t, c, \varepsilon)$ of the BVP (1), (2) with $x(0, c, \varepsilon) = c$ where $c \in B_k \subset \mathbb{R}^2$ and $\mid \varepsilon \mid < \varepsilon_0$ for some $\varepsilon_0 > 0$.

Proof Let

$$\Phi_0^1(c_1, c_2) = p_1(c_1, c_2) + q_1(c_1, c_2)$$
$$\Phi_0^2(c_1, c_2) = p_2(c_1, c_2) + q_2(c_1, c_2)$$

where $p_1(c_1, c_2)$ is a polynomial homogeneous of degree m in c_1 and c_2, $p_2(c_1, c_2)$ is a polynomial homogeneous of degree n in c_1 and c_2, and $q_i(c_1, c_2)$ consists of the term $kc_1^{l_1(i)}c_2^{l_2(i)}$ where $l_1^{(i)} + l_2^{(i)} < \min(m,n) = p$ for $i = 1, 2$. Thus we define ψ_0 be the mapping defined by

$$\psi_0(c_1, c_2) \to (p_1(c_1, c_2), p_2(c_1, c_2)).$$

If p_1 and p_2 have no common real linear factors then $d(\psi_0, B_k, 0)$ is defined for B_k, a ball of arbitrary radius. Define the homotopy

$$H(c, \lambda) = p(c_1, c_2) + \lambda q(c_1, c_2),\ \ 0 \le \lambda \le 1.$$

Let $m_0 < \min\{m, n\} = p$. By assumption on q_i,

$$\mid q_i(c_1, c_2) \mid < l \mid c \mid^{m_0},$$

for $\mid c \mid \ge k_0 > 0$ and $l > 0$. Notice that for $c \in \partial B_k$ we have $\mid \Phi_0(c) \mid > 0$. Thus we choose $k > 0$ such that

$$\begin{aligned} \mid \Phi_0(c) \mid &\ge \mid p(c_1, c_2) \mid - \mid q(c_1, c_2) \mid \\ &> d \mid c \mid^p - l \mid c \mid^{m_0} \\ &= \mid c \mid^p (d - l \mid c \mid^{m_0 - p}) \\ &> 0. \end{aligned}$$

We have $0 <\mid c \mid< (\frac{d}{l})^{p-m_0} = k$. Therefore if $c \in \partial B_k$ then $\mid H(c, \lambda) \mid > 0$. This means we assume that the terms of highest degree of p_1 and p_2 have no common real linear factors after reduction using the conditions 1, 2, 3, and 4 above. Thus $0 \notin H(\partial B_k, \lambda)$ for $0 \le \lambda \le 1$ and therefore for sufficiently large B_k

$$d(\Phi_0, B_k, 0) = d(\psi_0, B_k, 0) \neq 0,$$

where $\psi_0 = (p_1, p_2)$. Thus, by the invariance under homotopy of degree

$$H_\varepsilon = \lambda\Phi_0 + (1 - \lambda)\Phi_\varepsilon,\ \ \lambda \in [0, 1]$$

for sufficiently large B_k and sufficiently small ε, $\mid \varepsilon \mid< \varepsilon_0$. Hence there is a solution $x(t, c, \varepsilon)$ of the BVP (1), (2) with $x(0, c, \varepsilon) = c$ where $c \in B_k \subset \mathbb{R}^2$ and $\mid \varepsilon \mid< \varepsilon_0$ for some $\varepsilon_0 > 0$, provided solutions $x(t, c, \varepsilon)$ of initial value problems exist on [0,1] for each (c, ε).

Remark 3.3. In this paper, we find that an arbitrary small change in $A(t)$ will affect the structure of the set of solutions which change completely. We also find that the value of the local degree depends on the function $f(t, y, y', \varepsilon)$.

4 Applications and Examples

In this section, we apply the results of paper [4] to the problem of finding the solutions of three-point BVPs to systems of first-order equations. First we study the degenerate case for $\alpha = \sqrt{2}$ in the interval $[0, 2\pi]$ with rank $\mathcal{L}_{\alpha=\sqrt{2}} = 1 < 2$. Then we study the totally degenerate case, rank $\mathcal{L} = 0$ for general boundary conditions and give an example where Borsuk's Theorem or Theorem 3.2 applies. We will use the following facts in solving the examples.

$$\int_0^{1/2} \sin^n 4\pi s \cos^m 4\pi s \; ds \neq 0,$$
$$\int_0^{1} \sin^n 4\pi s \cos^m 4\pi s \; ds \neq 0 \tag{12}$$

if and only if both n and m are even.

$$\int_0^{1} \sin^n 2\pi s \cos^m 2\pi s \; ds \neq 0 \tag{13}$$

if and only if both n and m are even.

Rank $\mathcal{L}_{\alpha=\sqrt{2}} = 1 < 2$, $\alpha = \sqrt{2}$ and $y'(0) = 0$.

Consider

$$y'' + y = \varepsilon f(t, y, y', \varepsilon), \quad t \in [0, 2\pi], \tag{14}$$
$$y(2\pi) = \sqrt{2} y(\pi/4), \; y'(0) = 0, \tag{15}$$

where $\eta = \pi/4$, $\alpha = \sqrt{2}$ and $f \in C([0,1] \times \mathbb{R}^2 \times (-\varepsilon_0, \varepsilon_0); \mathbb{R})$. Then (14), (15) is equivalent to

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ f(t, x_1, x_2, \varepsilon) \end{pmatrix}, \tag{16}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(\pi/4) \\ x_2(\pi/4) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(2\pi) \\ x_2(2\pi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{17}$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $N = \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}$, $R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $F(t, x, \varepsilon) = \begin{pmatrix} 0 \\ f(t, x_1, x_2, \varepsilon) \end{pmatrix}$. We obtain $Y(t) = e^{At} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$,
$Y_0(t) = Y(t) Y^{-1}(0) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$,
$Y_0(2\pi) = \begin{pmatrix} \cos 2\pi & \sin 2\pi \\ -\sin 2\pi & \cos 2\pi \end{pmatrix}$, $Y_0(\pi/4) = \begin{pmatrix} \cos \pi/4 & \sin \pi/4 \\ -\sin \pi/4 & \cos \pi/4 \end{pmatrix}$ and

$Y(t)Y^{-1}(s) = e^{A(t-s)} = \begin{pmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{pmatrix}$. Then by Lemma 2.2, solving the problem (16), (17) is reduced to that of solving $\mathcal{L}_{\alpha=\sqrt{2}}c = \varepsilon\mathcal{N}(c,\alpha,\eta,\varepsilon) + d$ for c. Thus we find $\mathcal{L}_{\alpha=\sqrt{2}}$ and $\mathcal{N}(c,\alpha,\eta,\varepsilon)$.

$$\begin{aligned}
\mathcal{L}_{\alpha=\sqrt{2}} &= M + NY_0(\pi/4) + RY_0(2\pi) \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{N}(c,\alpha,\eta,\varepsilon) &= -\int_0^{\pi/4} \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} e^{A(\frac{\pi}{4}-s)} F(s, x(s,c,\varepsilon),\varepsilon)\, ds \\
&\quad - \int_0^{2\pi} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e^{A(2\pi-s)} F(s, x(s,c,\varepsilon),\varepsilon)\, ds \\
&= (\mathcal{N}_1(c,\alpha,\eta,\varepsilon), 0);
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{N}_1(c,\alpha,\eta,\varepsilon) &= \int_0^{\pi/4} \sqrt{2}\sin(\pi/4 - s) f(s, x_1(s,c,\varepsilon), x_2(s,c,\varepsilon),\varepsilon) ds \\
&\quad - \int_0^{2\pi} \sin(2\pi - s) f(s, x_1(s,c,\varepsilon), x_2(s,c,\varepsilon),\varepsilon) ds,
\end{aligned}$$

and $d = 0$. Thus we have rank $(\mathcal{L}_{\alpha=\sqrt{2}}) = 1$. Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a basis for $Ker(\mathcal{L}_{\alpha=\sqrt{2}})$, and $Ker(\mathcal{L}_{\alpha=\sqrt{2}})$=Span e_1. Let P_1 be the matrix projection onto $Ker(\mathcal{L}_{\alpha=\sqrt{2}})$,
$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. So $P_2 = I - P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Set $H = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ so that $H\mathcal{L}_{\alpha=\sqrt{2}} = P_2$. Since $d = 0$, it follows that $P_1Hd = 0$. Therefore a necessary condition of Theorem 2.4 is satisfied. Then we apply Theorem 3.2. In order to study Φ_0, we must first obtain $x(t,c,0)$, that is the solution of $x' = A(t)x$. By Lemma 2.1, $x' = A(t)x$ has a solution $x(t)$ with $x(0) = c = (c_1, 0)^T$, where $x_2(0) = 0 = c_2$. Thus (14), (15) has a solution if $\varepsilon = 0$ namely $x_1(t,c,0) = c_1\cos t, x_2(t,c,0) = -c_1\sin t$. We compute

$$\begin{aligned}
P_1H\mathcal{N}(c,\alpha,\eta,\varepsilon) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathcal{N}_1(c,\alpha,\eta,\varepsilon) \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{N}_1(c,\alpha,\eta,\varepsilon) \\ 0 \end{pmatrix}.
\end{aligned}$$

Thus $\Phi_\varepsilon(c_1) = \mathcal{N}_1(c^1, \alpha, \eta, \varepsilon)$, where $P_2c = c^2 = \binom{0}{c_2}$ and $P_1c = c^1 = \binom{c_1}{0}$. Setting $\varepsilon = 0$, we have $\Phi_0(c_1) = \mathcal{N}_1(c^1, \alpha, \eta, 0)$, where $c^2(c^1, 0) = P_2Hd = 0$. Hence

$$\begin{aligned}\Phi_0(c_1) &= \int_0^{\pi/4} \sqrt{2}\sin(\pi/4 - s) f(s, c_1\cos\ s, -c_1\sin s, 0)ds \\ &\quad - \int_0^{2\pi} \sin(2\pi - s) f(s, c_1\cos s, -c_1\sin s, 0)ds.\end{aligned}$$

Now we state an example where the value of the local degree depends on the function $f(t, y, y', \varepsilon)$.

Example 1

In system (16), let $f(t, x_1, x_2, \varepsilon) = ax_1^3 + bx_2$ so that $f \in C([0, 2\pi] \times \mathbb{R}^2 \times (-\varepsilon_0, \varepsilon_0); \mathbb{R})$. We have $x_1(t, c, 0) = c_1\cos t$, $x_2(t, c, 0) = -c_1\sin t$. Thus $f(t, c_1\cos\ t, -c_1\sin\ t, 0) = ac_1^3\ \cos^3 t - bc_1\sin t$. Using condition (13), we obtain

$$\begin{aligned}\Phi_0(c_1) &= \int_0^{\pi/4} \sqrt{2}\sin(\pi/4 - s)(ac_1^3\cos^3 s - bc_1\sin s)ds \\ &\quad - \int_0^{2\pi} \sin(2\pi - s)(ac_1^3\cos^3 s - bc_1\sin s)ds \\ &= \int_0^{\pi/4} \{(\cos\ s - \sin s)(ac_1^3\cos^3 s - bc_1\sin s)\}ds - \int_0^{2\pi} bc_1\sin^2 sds \\ &= \int_0^{\pi/4} \{ac_1^3\cos^4 s - bc_1\cos s\sin\ s - ac_1^3\sin s\cos^3 s + bc_1\sin^2 sds - bc_1\pi\}ds \\ &= ac_1^3(\frac{3\pi}{32} + \frac{1}{16}) - bc_1(\frac{7\pi}{8} + \frac{1}{2}).\end{aligned}$$

Since $\Phi_0(c_1)$ is odd, the local degree is odd and therefore nonzero. Then for sufficiently large B_k and sufficiently small ε, $d(\Phi_\varepsilon, B_k, 0) = d(\Phi_0, B_k, 0) \neq 0$.

Next we apply Borsuk's Theorem in Example 2, and then Theorem 3.2 in Example 3 to find the local degree of a mapping in Euclidean 2-space defined by homogeneous polynomials

Rank $\mathcal{L} = 0$.

Consider

$$y'' + 16\pi^2 y = \varepsilon f(t, y, y', \varepsilon), \quad t \in [0, 1], \tag{18}$$

$$2y(0) - y(1/2) - y(1) = 0, \tag{19}$$

$$-y'(1/2) + y'(1) = 0, \tag{20}$$

where $\eta = 1/2 \in (0,1)$, $f \in C([0,1] \times \mathbb{R}^2 \times (-\varepsilon_0, \varepsilon_0); \mathbb{R})$. This can be written in matrix form as a system of first-order equations.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -16\pi^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ f(t, x_1, x_2, \varepsilon) \end{pmatrix} \tag{21}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1(1/2) \\ x_2(1/2) \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{22}$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -16\pi^2 & 0 \end{pmatrix}$, $M = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $N = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$,
$R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. $Y(t) = e^{At} = \begin{pmatrix} \cos 4\pi t & \sin 4\pi t/(4\pi) \\ -4\pi \sin 4\pi t & \cos 4\pi t \end{pmatrix}$,
$Y^{-1}(t) = \begin{pmatrix} \cos 4\pi t & -\sin 4\pi t/(4\pi) \\ 4\pi \sin 4\pi t & \cos 4\pi t \end{pmatrix}$,
$Y_0(t) = Y(t)Y^{-1}(0) = \begin{pmatrix} \cos 4\pi t & \sin 4\pi t/(4\pi) \\ -4\pi \sin 4\pi t & \cos 4\pi t \end{pmatrix}$, $Y_0(1/2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $Y_0(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then by Lemma 2.2, the problem of solving (21), (22) is reduced to that of solving $\mathcal{L}c = \varepsilon \mathcal{N}(c, \alpha, \eta, \varepsilon) + d$ for c. Thus we find $\mathcal{L}$ and $\mathcal{N}(c, \alpha, \eta, \varepsilon)$.

$$\begin{aligned} \mathcal{L} &= M + NY_0(1/2) + RY_0(1) \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus we have rank $\mathcal{L} = 0$. Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, is a basis for $Ker(\mathcal{L})$, and $Ker(\mathcal{L}) = Span(e_1, e_2)$. Let P_1 be the matrix projection onto $Ker(\mathcal{L})$, $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So $P_2 = I - P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Set $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so that $H\mathcal{L} = P_2$.

We obtain

$$\begin{aligned}
\mathcal{N}(c,\alpha,\eta,\varepsilon) &= -\int_0^{1/2}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos 4\pi s & -\sin 4\pi s/(4\pi)\\ 4\pi\sin\ 4\pi s & \cos 4\pi s\end{pmatrix}\\
&\quad\times\begin{pmatrix}0\\ f(s,x_1(s,c,\varepsilon),x_2(s,c,\varepsilon),\varepsilon)\end{pmatrix}ds\\
&\quad-\int_0^{1}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos 4\pi s & -\sin 4\pi s/(4\pi)\\ 4\pi\sin 4\pi s & \cos 4\pi s\end{pmatrix}\\
&\quad\times\begin{pmatrix}0\\ f(s,x_1(s,c,\varepsilon),x_2(s,c,\varepsilon),\varepsilon)\end{pmatrix}ds\\
&= \int_0^{1/2}\begin{pmatrix}-\sin\ 4\pi s/(4\pi)\\ \cos\ 4\pi s\end{pmatrix}f(s,x_1(s,c,\varepsilon),x_2(s,c,\varepsilon),\varepsilon))ds\\
&\quad+\int_0^{1}\begin{pmatrix}-\sin\ 4\pi s/(4\pi)\\ -\cos\ 4\pi s\end{pmatrix}f(s,x_1(s,c,\varepsilon),x_2(s,c,\varepsilon),\varepsilon))ds\\
&= \begin{pmatrix}\mathcal{N}_1(c,\alpha,\eta,\varepsilon)\\ \mathcal{N}_2(c,\alpha,\eta,\varepsilon)\end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{N}_1(c,\alpha,\eta,\varepsilon) &= -\int_0^{1/2}\sin 4\pi s/(4\pi) f(s,x_1(s,c,\varepsilon),x_2(s,c,\varepsilon),\varepsilon))ds\\
&\quad-\int_0^{1}\sin 4\pi s/(4\pi) f(s,x_1(s,c,\varepsilon),x_2(s,c,\varepsilon),\varepsilon))ds,
\end{aligned}$$

$$\mathcal{N}_2(c,\alpha,\eta,\varepsilon) = -\int_{1/2}^{1}\cos 4\pi s f(s,x_1(s,c,\varepsilon),x_2(s,c,\varepsilon),\varepsilon))ds,$$

and $d=0$. Since $d=0$, it follows that $P_1Hd=0$. Therefore a necessary condition of Theorem 2.4 is satisfied. Then we apply Theorem 3.2. In order to study Φ_0, we must first obtain $x(t,c,0)$, that is the solution of $x'=A(t)x$. By Lemma 2.1, $x'=A(t)x$ has a solution $x(t)$ with $x(0)=c=(c_1,c_2)^T$. Thus (18), (19), (20) has a solution if $\varepsilon=0$ namely $x_1(t,c,0)=c_1\cos 4\pi t + c_2\sin 4\pi t/(4\pi), x_2(t,c,0)=-4\pi c_1\sin 4\pi t + c_2\cos 4\pi t$. We compute

$$P_1H\mathcal{N}(c,\alpha,\eta,\varepsilon) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\mathcal{N}_1(c,\alpha,\eta,\varepsilon)\\ \mathcal{N}_2(c,\alpha,\eta,\varepsilon)\end{pmatrix}.$$

Thus

$$\Phi_\varepsilon(c_1,c_2) = \begin{pmatrix}\mathcal{N}_1(c,\alpha,\eta,\varepsilon)\\ \mathcal{N}_2(c,\alpha,\eta,\varepsilon)\end{pmatrix}.$$

Setting $\varepsilon=0$, we have

$$\Phi_0(c_1,c_2) = \begin{pmatrix}\mathcal{N}_1(c,\alpha,\eta,0)\\ \mathcal{N}_2(c,\alpha,\eta,0)\end{pmatrix}.$$

Example 2
In system (21), let $f(t, x_1, x_2, \varepsilon) = x_2^3$ so that $f \in C([0, 2\pi] \times \mathbb{R}^2 \times (-\varepsilon_0, \varepsilon_0); \mathbb{R})$. Then

$$f(t, c_1 \cos 4\pi t + c_2 \sin 4\pi t/(4\pi), -4\pi c_1 \sin 4\pi t + c_2 \cos 4\pi t, 0) = -64\pi^3 c_1^3 \sin^3 4\pi t \\ +48\pi^2 c_1^2 c_2 \sin^2 4\pi t \cos 4\pi t - 12\pi c_1 c_2^2 \sin 4\pi t \cos^2 4\pi t + c_2^3 \cos^3 4\pi t.$$

Using condition (12), we obtain

$$\begin{aligned}
& \Phi_0^1(c_1, c_2) \\
= & -\int_0^{1/2} \{\frac{\sin 4\pi s}{4\pi} f(s, c_1 \cos 4\pi s + c_2 \sin 4\pi s/(4\pi), -4\pi c_1 \sin 4\pi s + c_2 \cos\ 4\pi s, 0)\} ds \\
& -\int_0^{1} \{\frac{\sin 4\pi s}{4\pi} f(s, c_1 \cos 4\pi s + c_2 \sin 4\pi s/(4\pi), -4\pi c_1 \sin 4\pi s + c_2 \cos\ 4\pi s, 0)\} ds \\
= & -\int_0^{1/2} \{16\pi^2 c_1^3 \sin^4 4\pi s + 3c_1 c_2^2 \sin^2 4\pi s \cos^2 4\pi s\} ds \\
& -\int_0^{1} \{16\pi^2 c_1^3 \sin^4 4\pi s + 3c_1 c_2^2 \sin^2 4\pi s \cos^2 4\pi s\} ds \\
= & \; 9\pi^2 c_1^3 + \frac{9c_1 c_2^2}{16}
\end{aligned}$$

and

$$\begin{aligned}
& \Phi_0^2(c_1, c_2) \\
= & -\int_{1/2}^{1} \cos 4\pi s \{f(s, c_1 \cos 4\pi s + c_2 \sin 4\pi s/(4\pi), -4\pi c_1 \sin 4\pi s + c_2 \cos\ 4\pi s, 0)\} ds \\
= & -\int_0^{1/2} \{48\pi^2 c_1^2 c_2 \sin^2 4\pi s \cos^2 4\pi s + c_2^2 \cos^4 4\pi s\} ds \\
= & -3\pi^2 c_1^2 c_2 + \frac{3c_2^3}{16\pi}.
\end{aligned}$$

Since $\Phi_0(c_1, c_2) = (\Phi_0^1(c_1, c_2), \Phi_0^2(c_1, c_2))$ is continuous, odd on ∂B_k and $0 \notin \Phi_0(\partial B_k)$, the local degree is odd and therefore nonzero. Then for sufficiently large B_k and sufficiently small ε, $d(\Phi_\varepsilon, B_k, 0) = d(\Phi_0, B_k, 0) \neq 0$.

Example 3
In system (21), let $f(t, x_1, x_2, \varepsilon) = x_1^2 \cos 4\pi t + x_2 \cos^2 4\pi t + x_1 \sin^2 4\pi t$ so that $f \in C([0, 2\pi] \times \mathbb{R}^2 \times (-\varepsilon_0, \varepsilon_0); \mathbb{R})$. Then

$$f(t, c_1 \cos\ 4\pi t + c_2 \sin 4\pi t/(4\pi), -4\pi c_1 \sin 4\pi t + c_2 \cos 4\pi t, 0) = c_1^2 \cos^3 4\pi t \\ +\frac{c_1 c_2}{2\pi} \cos^2 4\pi t \sin 4\pi t + \frac{c_2^2 \cos 4\pi t \sin^2 4\pi t}{16\pi^2} - 4\pi c_1 \cos^2 4\pi t sin 4\pi t \\ +c_2 \cos^3 4\pi t + c_1 \sin^2 4\pi t \cos 4\pi t + \frac{c_2}{4\pi} \sin^3 4\pi t.$$

Using condition (12), we obtain

$$\begin{aligned}
&\Phi_0^1(c_1,c_2)\\
= & -\int_0^{1/2}\{\frac{\sin 4\pi s}{4\pi}f(s, c_1\cos 4\pi s + c_2\sin 4\pi s/(4\pi), -4\pi c_1\sin 4\pi s + c_2\cos\ 4\pi s, 0)\}ds\\
& -\int_0^{1}\{\frac{\sin 4\pi s}{4\pi}f(s, c_1\cos 4\pi s + c_2\sin 4\pi s/(4\pi), -4\pi c_1\sin 4\pi s + c_2\cos\ 4\pi s, 0)\}ds\\
= & -\int_0^{1/2}\{[\frac{c_1c_2}{8\pi^2}-c_1]\cos^2 4\pi s\sin^2 4\pi s + \frac{c_2\sin^4 4\pi s}{16\pi^2}\}ds\\
& -\int_0^{1}\{[\frac{c_1c_2}{8\pi^2}-c_1]\cos^2 4\pi s\sin^2 4\pi s + \frac{c_2\sin^4 4\pi s}{16\pi^2}\}ds\\
= & \frac{-3c_1c_2}{128\pi^2}+\frac{3c_1}{16}-\frac{9c_2}{64\pi^2}
\end{aligned}$$

and

$$\begin{aligned}
&\Phi_0^2(c_1,c_2)\\
= & -\int_{1/2}^{1}\{\cos 4\pi s f(s, c_1\cos 4\pi s + c_2\sin 4\pi s/(4\pi), -4\pi c_1\sin 4\pi s + c_2\cos\ 4\pi s, 0)\}ds\\
= & -\int_0^{1/2}\{(c_1^2+c_2)\cos^4 4\pi s + (c_2^2+c_1)\cos^2 4\pi s\sin^2 4\pi s\}ds\\
= & -(\frac{3\pi c_1^2}{4}+\frac{c_2^2}{256\pi^2}) - (\frac{3\pi c_2}{4}+\frac{c_1}{16}).
\end{aligned}$$

Let

$$\begin{aligned}
\Phi_0^1(c_1,c_2) &= p_1(c_1,c_2) + q_1(c_1,c_2)\\
\Phi_0^2(c_1,c_2) &= p_2(c_1,c_2) + q_2(c_1,c_2)
\end{aligned}$$

where

$$\begin{aligned}
p_1(c_1,c_2) &= \tfrac{-3c_1c_2}{128\pi^2}, \quad q_1 = \tfrac{3c_1}{16} - \tfrac{9c_2}{64\pi^2},\\
p_2(c_1,c_2) &= -(\tfrac{3\pi c_1^2}{4} + \tfrac{c_2^2}{256\pi^2}), \quad q_2 = -(\tfrac{3\pi c_2}{4} + \tfrac{c_1}{16}).
\end{aligned}$$

Hence $p_1(c_1,c_2)$ is a polynomial homogeneous of degree $m = 2$ in c_1 and c_2, $p_2(c_1,c_2)$ is a polynomial homogeneous of degree $n = 2$ in c_1 and c_2, and $q_i(c_1,c_2)$ consists of the term $kc_1^{l_1(i)}c_2^{l_2(i)}$ where $l_1^{(i)} + l_2^{(i)} = 1 < \min(m,n) = 2$ for $i = 1, 2$. Thus we define ψ_0 to be the mapping defined by

$$\psi_0(c_1,c_2) \to (p_1(c_1,c_2), p_2(c_1,c_2)).$$

Since p_1 and p_2 have no common real factors, then $d(\psi_0, B_k, 0)$ is defined for B_k of arbitrary radius. After reduction using the conditions 1 and 4 in Theorem

3.2, ψ_0 is a constant. Hence $d(\psi_0, B_k, 0) = 0$. If the radius of B_k is sufficiently large then $d(\Phi_0, B_k, 0) = d(\psi_0, B_k, 0)$. Hence for sufficiently large B_k and sufficiently small ε, $d(\Phi_\varepsilon, B_k, 0) = 0$. Do the solutions exist? The answer is yes, $y \equiv 0$ for each $\varepsilon < \varepsilon_0$, in fact this is the only solution of the BVP (18), (19), (20).

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