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SOME STRUCTURE PROPERTIES OF UPPER Q-FUZZY INDEX ORDER WITH UPPER Q- FUZZY SUBGROUPS

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Abstract: In this paper we shall study some properties for upper Q- fuzzy subgroups, some lemma and theorem for this subject. We shall study the upper Q- fuzzy index with the upper fuzzy sub groups; also we shall give some new definitions for this subject. On the other hand we shall give the definition of the upper normal fuzzy subgroups, and study the main theorem for this. We shall also give new results on this subject.

Key words: Fuzzy set, upper Q- fuzzy subgroups, upper normal Q- fuzzy subgroups, upper Q- fuzzy order. upper Q-fuzzy cossets, upper Q-fuzzy index.
AMS subject classification (2000) : 20N25

SECTION-1 INTRODUCTION : Zadeh′s classical paper [16] of 1965 introduced the concepts of fuzzy sets and fuzzy set operations. Foster [5] combined the structure of a fuzzy topological space. The study of the fuzzy algebraic structures started with the introduction of concepts of fuzzy subgroups and fuzzy (left, right) ideals in the pioneering paper of Rosenfeld [11].Anthony and Sherwood [1] redefined fuzzy subgroups using the concept of triangular norm. Several mathematicians[4,6,8] followed the Rosenfeld approach in investigating fuzzy algebra where given ordinary algebraic structure on a given set X is assumed then introducing the fuzzy algebraic structure as a fuzzy subset A of X satisfying some suitable conditions. In this paper we investigate further the theory of upper fuzzy subgroups and the upper fuzzy index. We also add some result on this subject. Throughout this paper, G will denote a group and "e" will denote its identity element. Let sup, inf, card, min, max will denote the supermum, infimum, cardinality, minimum, maximum respectively. To know more of this subject, it is possible to return to Doctorate thesis of Mourad Massa'deh (Damascus University 2008).The concept of a fuzzy subset was introduced by Zadeh (1965). Fuzzy subgroup and its important properties were defined and established by Rosenfeld (1971). There are many authors who have studied about it. After this time it was necessary to define upper fuzzy subgroups and upper normal fuzzy subgroups. The notion of the definition of an upper fuzzy subgroup was introduced by Rosenfeld. Many researchers Abd – Allahetal. (1996),Chengyi (1998), Dib and Hassan (1998), Tang and Zhang (2001), Syransu and Ruy (1998), Massa'deh(2008) studied the properties of groups and subgroups by the definition of fuzzy sub groups. A.Solairaju and R.Nagarajan [14]introduced the notion of Q- fuzzy groups.The aim of this paper is to study and prove some properties and theorem for upper Q- fuzzy order. In this paper, G is a group with identity e. Z, N is the integer, and natural number respectively.

SECTION-2 Preliminaries

2.1 Definition: let X be a set. A fuzzy set A of X is just a function $A : X \rightarrow [0, 1]$

2.2 Definition: let G be a group and μ be a fuzzy set on G. A is said to be an upper fuzzy subgroup of G, if for all $x, y \in G$ (i) $A(xy) \leq \max\{A(x), A(y)\}$ (ii) $A(x^{-1}) = A(x)$

2.3 Definition : An upper fuzzy subgroup A of a group G is called upper normal fuzzy subgroups ifA $(x^{-1}yx) \leq A(y)$, for all $x, y \in G$.(Equivalently, $A(xyx^{-1}) = A(y)$ for all $x,y \in G$ (Equivalently, $A(xy) = A(yx)$ for all $x,y \in G$).

2.4 Definition: Let λ be an upper fuzzy subgroup of G. For any $x \in G$, the smallest positive integer n such that $\lambda(x^n) = \lambda(e)$ is called an upper fuzzy order of x. If there does not exist such n then x is said to have an infinite we upper fuzzy order. We shall denote the upper fuzzy order of x by $O(\lambda(x))$.

Example Let G = { e , a , b , ab } be the Klein four group and let A = { (e , 1/4) , (a , 3 /4) , (b , 3/4) , (ab , 1/4) } be an upper Q-fuzzy subgroup, then O (A (ab,q)) = 1 and O (A (a,q)) = 2.

2.5Definition: Let Q and G be a set and a group respectively. A mapping A: G×Q → [0,1] is called Q-fuzzy set in G. If a Q- fuzzy set A is called upper Q- fuzzy subgroup of G if (QFG1) $A(xy,q) \leq \max \{ A(x,q) , A(y,q)\}$ (QFG2) $A(x^{-1},q) = A(x,q)$ (QFG3) $A(e,q) = 1$ for all x,yε G and qεQ.

2.6 Definition: A upper Q-fuzzy subgroup A_H of G is called a upper Q- fuzzy normal subgroup of G if $A_H(xy,q) = A_H(yx,q)$ for all x,y of G and q ε Q.

2.7 Definition Let A be an upper Q-fuzzy subgroup of a group G.A Q-fuzzy subset satisfying: $[gA] (x,q) = A (g^{-1}x,q)$ for all x, g ε G; is called an upper fuzzy left cosets of A, while that satisfies: $[Ag] (x,q) = A(xg^{-1},q)$ for all x, g ε G; is called an upper fuzzy right cosets of A.
In the following study, the upper Qfuzzy left and right cosets must be fuzzy sets but not necessarily be upper Q-fuzzy subgroups.

SECTION-3 Properties of Upper Q- fuzzy index order with fuzzy subgroups

3.1 Proposition: Let A be an upper Q-fuzzy subgroup of the group G. Then for any integer n and x ε G, we have $A (x^n .q) \leq A (x,q)$.

3.2 Proposition: Let G be a group and let A be an upper Q-fuzzy subgroup of the group G; let x ε G be of finite order k ; if r ε N and k are relatively prime, then $A (x^r ,q) = A (x,q)$.

Proof :-

Since r, k is relatively prime, then by Bizout Theorem There exists a, b ε Z such that 1 = ar + bk ; therefore $A (x,q) = A (x^{ar + bk} ,q) = A((x^{ar}.x^{bk}),q) = A (x^{ar} ,q) \leq A (x^r,q) \leq A (x,q)$. Then we get $A (x,q) \leq A (x r,q) \leq A (x,q)$,therefore $A (xr,q) = A (x,q)$.

3.3 Lemma: Let A be an upper Q- fuzzy subgroup of the group G, for x ε G. If A (e,q) = $A (x^n ,q)$ for some n ε Z, then O (A (x, q)) divides n .

Proof :-

Suppose that $O(A(x,q)) = k$, then by Euclidean Algorithm , there exists $a, b \varepsilon Z$. Such that $n = ka + b; 0 \le b < k$; thus $A(x^{b},q) = A(x^{n-ka},q) = A(x^{n}(x^{k})^{-a},q) \le \max\{A(x^{n},q), A((x^{k})^{-a},q\} \le \max\{A(e,q), A(x^{k},q)\} = \max\{A(e,q), A(e,q)\} = A(e,q)$.Thus $A(x^{b},q) \le A(e,q)$; also $A(x^{b},q) \ge A(e,q)$, then we get $A(x^{b},q) = A(e,q)$.Then $b = 0$; also $n = ka$ which is given $O(A(x,q))$ divides n.

3.4 Proposition: Let A be an upper Q- fuzzy subgroup of the group G, and let $O(A(x,q)) = k$, such that $x \varepsilon G$. If $t \varepsilon Z$ with $d = (t, k)$, then $O(A(x^{t},q)) = k/d$.

Proof :-

Suppose that $O(A(x^{t},q)) = n$, we get $A((x^{t},q)^{k/d}) = A((x^{t})^{ka},q) = A(x^{ka},q)$; for some integer a $\le A(x^{k},q) = A(e,q)$

By Lemma 3.3 ; $n/ k/d$ and $d = (t, k)$Then there exists $b, c \varepsilon Z$ such that $bt + ck = d$; therefore $A(x^{nd},q) = A(x^{n(bt+ck)},q) = A((x^{nbt}. x^{nck}), q) \le \max\{A((x^{b},q)^{nt}), A((x^{c},q)^{nk}\} = \max\{A((x^{b},q)^{nt}), A(((x^{c},q)^{n})^{k}\} \le \max\{A(x^{b},q), A(x^{c},q)^{k}\} \le \max\{A(e,q), A(e,q)\} = A(e,q)$Therefore k/nd by lemma 3.3; this means that $k/d\ /n$, consequently $n = k/d$.

3.5 Proposition: Let A be an upper Q-fuzzy subgroup of the group G, let $O(A(x,q)) = A$; $x \varepsilon G$. If $k \varepsilon Z$ such m, n relatively prime, then $A(x^{k},q) = A(x,q)$.

Proof :-

Since m, k relatively prime, $(m, k) = 1$Then there exists $a, b \varepsilon Z$ such that $ma + kb = 1$

$$\begin{aligned} A(x,q) &= A(x^{ma+kb},q) = A((x^{m})^{a}.(x^{k})^{b},q) \\ &\le \max\{A(x^{m},q)^{a}, A(x^{k},q)^{b}\} \\ &\le \max\{A(x^{m},q), A(x^{k},q)\} \\ &\le \max\{A(e,q), A(x^{k},q)\} \\ &= A(x^{k},q) \\ &\le A(x,q) \end{aligned}$$

Therefore $A(x^{k},q) = A(x,q)$.

3.6 Lemma: Let A be an upper Q- normal fuzzy subgroup of the group G. Then $O(A(x,q)) = O(A(y^{-1}xy,q))$ for all $x, y \varepsilon G$.

Proof :-

Let $x, y \varepsilon G$, then we have $A(x^{m}) = A(y^{-1} x m y) = A((y^{-1}xy)^{m})$, for all $m \varepsilon Z$. Thus $O(A(x,q)) = O(A(y^{-1}xy),q)$.

3.7 Remark: If A is not upper normal Q- fuzzy subgroup of the group G, then above lemma3.6 is not true.

Example: Let G be the Dihedral group

G = { e , a , b , a 2 , ab , ba } and let A = { (e , 1/ 5) , (a , 4 / 5) , (b , 1 / 2) , (a 2 , 4/ 5) , (ab , 4 / 5) , (ba , 4 / 5) }Since O (A (b)) = 1 ≠ 2 = O (A (a^{-1} b a)).

3.8 Proposition: Let G be a finite group and let A, λ be an upper Q-fuzzy subgroups, if λ Í m and A (e ,q) = λ (e, q) .Then O (A (x, q)) / O (λ (x, q)) for all x ε G such that O (λ (x ,q)) is finite.

Proof :-

Suppose that O (λ (x,q)) = k , then A (e,q) = λ (e,q) = λ (x^n ,q) ≤ A (x^n ,q) , since A (e ,q) ≤ A (x^n ,q) and A(e ,q) = A (x^n ,q) . Thus, O (A (x ,q)) / n (by lemma 3.3).

3.9 Lemma: Let A be an upper Q- fuzzy subgroup of the group G and let x , y ε G, such that (O (A (x ,q)) , O (A (y,q)))= 1 [that is, O (A (x,q)) , O (A (y,q)) relatively prime]and xy = yx . If A (e ,q) = A (xy,q) then A (x,q) and A (y ,q) =A (e ,q).

Proof :-

Suppose that O (A (x,q)) = k and O (A (y,q)) = m, then A(e,q) = A (xy,q) ≥ A ($(xy)^n$,q) = A (($x^n y^n$),q), since A(($x^n y^n$),q)= A (e,q) but A (x^n ,q) = A(($x^n y^n y^{-n}$),q) ≤ max { (A ($x^n y^n$),q), A (y^{-n} ,q) } = max { A ($x^n y^n$,q), A (y^n ,q) } = max { A (e ,q), A (e, q) } = A (e,q)Therefore A (x^n ,q) = A (y^n ,q) = A (e ,q), then k / n by lemma 3.3. Since k , m relatively prime [(k , m) = 1] thus k = 1 this means that A (x,q) = A (e,q); the same proof for A (y ,q) = A (e ,q).

3.10 Proposition: Let A be an upper Q-fuzzy subgroup of a cyclic group G and let x , y be two generators of , then O(A (x, q)) = O (A (y ,q)).

Proof:-

Suppose that G is a finite cyclic group with O(G) = m ,since G is generated by x and y. Then we get O(x) = O(y) = m, on the other hand y = x n ; n ε Z and we must have (m , n) = 1; thus O (A (x,q)) = O (A (x^n,q)) = O (A (y, q)) = n . Note if G is an infinite group, then y = x^{-1}

3.11 Lemma If A is an upper Q-fuzzy subgroup of G, then A (e,q) ≤ A(x,q) for every x ε G.

Proof:

A (e,q) = A ((x . x^{-1}),q) ≤ max {A (x,q), A (x^{-1},q)}

= max {A (x,q) ,A (x,q)} = A (x,q).

3.12 Proposition

Let G be a group and A be an upper Q-fuzzy subgroup of G. Then for any x, y ε G such that $A(x,q) \neq A(y,q)$, we have $A(xy,q) = \max\{A(x,q), A(y,q)\}$

Proof :

Assume that $A(y,q) > A(x,q)$, then :$A(y,q) = A((x^{-1}xy),q) \leq \max\{A(x^{-1},q), A(xy,q)\} = \max\{A(x,q), A(xy,q)\} = A(xy,q)$

Also A is an upper Q-fuzzy subgroup, then :

$A(xy,q) \leq \max\{A(x,q), A(y,q)\} = A(y,q)$. Thus $A(y,q) \leq A(xy,q) \leq A(y,q)$, this implies that $A(xy,q) = \max\{A(x,q), A(y,q)\}$The same way if $A(x,q) > A(y,q)$.

2.8 Definition: An upper Q-fuzzy subgroup A of a group G is called upper normal Q-fuzzy subgroup if $A((xyx^{-1}),q) \leq A(y,q)$, for all x , y ε G.

3.13 Proposition The following conditions are equivalent:

(I) G is an upper normal Q-fuzzy subgroup

(II) $A(xyx^{-1},q) = A(y,q)$ for all x,y ε G

(III) $A(xy,q) = A(yx,q)$ for all x,y ε G

Proof :

(I) → (II) :

$A(y,q) \geq A(xyx^{-1},q)$ for all x , y ε G, on the other hand we need to show $A(xyx^{-1},q) \geq A(y,q)$ for all x ,y ε G. $A(y,q) = A(x^{-1}xyx^{-1}x,q) \leq A(xyx^{-1},q)$. Then $A(y,q) \leq A(xyx^{-1},q)$, therefore $A(xyx^{-1},q) = A(y,q)$ for all x ,y ε G.

(II) → (III) :

Since $xy = x(yx)x^{-1}$, then $A(xy,q) = A(x(yx)x^{-1},q) = A(yx,q)$, then $A(xy,q) = A(yx,q)$ for all x ,y ε G.

(III) → (I):

Since $A(xyx^{-1},q) = A((xy)x^{-1},q) = A(x^{-1}(xy),q) = A(x^{-1}xy,q) = A(y,q)$,then $A(xyx^{-1},q) \leq A(y,q)$, for all x ,y ε G

Thus the conditions are equivalent.□

2.9 Definition : Let A be an upper Q-fuzzy subgroup of a group G. Then the upper Q-fuzzy index of A in G is defined by:

{ The set of the upper Q-fuzzy left cosets of A in G, that are themselves are upper Q-fuzzy subgroups the set of the upper fuzzy left cos ets of μ in G,[G:A] = Card

We shall call this type of the upper Q-fuzzy index by general upper Q-fuzzy index, because this type is not restricted by any conditions.

3.14 Proposition If A is an upper Q-fuzzy subgroup of G , then the value of the upper Q-fuzzy index is equal to the number of elements in A which is equal to A (e,q) adding for it one , i.e., [G : A] = card (element of A which is equal to A (e,q)) + 1).

Proof :

For all elements of A which is the value of it's elements equal to A(e,q).The upper Q-fuzzy left cosset of its elements are upper Q-fuzzy subgroups; also the upper Q-fuzzy left cosset of A (e,q) is also upper Q-fuzzy subgroup. And these satisfy the definition of the upper Q-fuzzy index. Therefore:

[G : A] = card (element of A which is equal to A (e,q) + 1).

3.15 Proposition: If A is an upper Q-fuzzy subgroup of a group G, and [G : A] = 3, then A is an upper normal Q-fuzzy subgroup of G.

Proof :

Suppose that A is an upper Q-fuzzy subgroup of G and [G : A] = 3,this means that A contains two elements having the same value of A(e,q) and let these two elements A (x,q), A (y,q) such that x, y ε G, we need to prove that A is an upper normal Q-fuzzy subgroup, which means that for all z ,h ε G, we need to show $A(z h z^{-1},q) \leq A(h,q)$ or $A(h z,q) = A(zh,q)$ or $A(h z h^{-1},q) \leq A(z,q)$ there exist many cases for A (h,q), A (z,q):

(i) If A (h,q) = A (z,q) and any of these is not equal to A (e,q), then

$$\begin{aligned} A(z h z^{-1},q) &\leq \max\{A(z h,q), A(z^{-1},q)\} \\ &= \max\{A(z h,q), A(z,q)\} \\ &\leq \max\{\max\{A(z,q), A(h,q)\}, A(z,q)\} \\ &= \max\{\max\{A(h,q), A(h,q)\}, A(h,q)\} \\ &= A(h,q) \end{aligned}$$

Then $A(z h z^{-1},q) \leq A(h,q)$.

(ii) If $A(h,q) \neq A(z,q)$ and any of these is not equal to A (e,q) by proposition 2.6.

$$A(h z,q) = \max\{A(h,q), A(z,q)\} \text{ and}$$

$A(z\,h,q) = \max\{A(z,q), A(h,q)\}$ thus

$A(z\,h,q) = \max\{A(z,q), A(h,q)\} = \max\{A(h,q), A(z,q)\} = A(h\,z,q)$, then $A(z\,h,q) = A(h\,z,q)$.

(iii) In this case we have two sub cases :

Sub case (a). In the case if the element h of G is the same of x or y or e it follows that $A(h,q) = A(e,q)$ and the element z of G is different from x, y and e then $A(e,q) \gneqq A(z,q)$ because there are only two elements in G and their values in A equal to $A(e,q)$, thus $A(h\,z\,h^{-1},q)$

$$\le \max\{A(h\,z,q), A(h^{-1},q)\} = \max\{A(h\,z,q), A(h,q)\}$$
$$\le \max\{\max\{A(h,q), A(z,q), A(h,q)\}$$
$$= \max\{A(z,q), A(h,q)\}$$
$$= A(z,q)$$

Then $A(h\,z\,h^{-1},q) \le A(z,q)$.

Subcase (b). In the case if the element z of G is the same of x or y or e it follows that $A(z,q) = A(e)$ and the element h of G is different from x, y and e then $A(e,q) \gneqq A(h,q)$ because there are only two elements in G and their values in μ equal to μ (e), thus $A(z\,h\,z^{-1},q) \le \max\{A(z\,h,q), A(z^{-1},q)\} = \max\{A(z\,h,q), A(z,q)\} \le \max\{\max\{A(z,q), A(h,q), A(z,q)\}$

$$= \max\{A(h,q), A(z,q)\} = A(h,q)$$

Then $A(z\,h\,z^{-1},q) \le A(h,q)$.

(iv) if the element h of G is the same of x or y or e it follows that $A(h,q) = A(e,q)$, and the element z of G is the same of x or y or e it follows that $A(z,q) = A(e,q)$ then $A(z\,h\,z^{-1},q) \le \max\{A(z\,h,q), A(z^{-1},q)\} = \max\{A(z\,h,q), A(z,q)\} \le \max\{\max\{A(z,q), A(h,q)\}, A(z,q)\}$

$$= \max\{\max\{A(h,q), A(h,q)\}, A(h,q)\} = A(h,q)$$

Then $A(z\,h\,z^{-1},q) \le A(h,q))$.

From the above, all cases that we have studied, we conclude that A is upper normal Q-fuzzy subgroup of G.

Conclusions

We have studied in this paper the definition of the upper Q-fuzzy order over an arbitrary group. Some proposition, lemma and examples given for it and this proposition and lemma are generalization for some properties in group theory.

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